Distance measures in cosmology

David W. Hogg

Institute for Advanced Study
Olden Lane, Princeton NJ 08540
hogg@ias.edu

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1 Introduction

In cosmology (or to be more specific, *cosmography,* the measurement of the Universe) there are many ways to specify the distance between two points, because in the expanding Universe, the distances between comoving objects are constantly changing, and Earth-bound observers look back in time as they look out in distance. The unifying aspect is that all distance measures somehow measure the separation between events on radial null trajectories, i.e., trajectories of photons which terminate at the observer.

In this note, formulae for many different cosmological distance measures are provided. I treat the concept of "distance measure" very liberally, so, for instance, the lookback time and comoving volume are both considered distance measures. The bibliography of source material can be consulted for derivations; this is merely a "cheat sheet." *C* routines (KR) which compute all of these distance measures are available from the author upon request. Comments and corrections are highly appreciated, as are acknowledgments in research that makes use of this summary or code.

2 Cosmographic parameters

The *Hubble constant* H_0 is the constant of proportionality between recession speed v and distance d in the expanding Universe;

$$v = H_0\, d \tag{1}$$

The subscripted "0" refers to the present epoch because in general H changes with time. The dimensions of H_0 are inverse time, but it is usually written

$$H_0 = 100\, h\ \mathrm{km\,s^{-1}\,Mpc^{-1}} \tag{2}$$

where h is a dimensionless number parameterizing our ignorance. The inverse of the Hubble constant is the *Hubble time* t_H

$$t_H \equiv \frac{1}{H_0} = 9.78\times 10^9\, h^{-1}\ \mathrm{yr} = 3.09\times 10^{17}\, h^{-1}\ \mathrm{s} \tag{3}$$

and the speed of light c times the Hubble time is the *Hubble distance* D_H

$$D_H \equiv \frac{c}{H_0} = 3000\, h^{-1}\ \mathrm{Mpc} = 9.26\times 10^{25}\, h^{-1}\ \mathrm{m} \tag{4}$$

These quantities set the scale of the Universe, and often cosmologists work in geometric units with $c = t_H = D_H = 1$.

The mass density ρ of the Universe and the value of the cosmological constant Λ are dynamical properties of the Universe, affecting the time evolution of the metric, but in these notes we will treat them as purely kinematic parameters. They can be made into dimensionless density parameters Ω_M and Ω_Λ by

$$\Omega_M \equiv \frac{8\pi\, G\, \rho_0}{3\, H_0^2} \tag{5}$$

$$\Omega_\Lambda \equiv \frac{\Lambda}{3\, H_0^2} \tag{6}$$

(Peebles, 1993, pp. 310–313), where the subscripted "0"s indicate that the quantities (which in general evolve with time) are to be evaluated at the present epoch. A third density parameter Ω_R measures the "curvature of space" and can be defined by the relation

$$\Omega_M + \Omega_\Lambda + \Omega_R = 1 \tag{7}$$

These parameters totally determine the geometry of the Universe if it is homogeneous, isotropic, and matter-dominated. By the way, the critical density $\Omega = 1$ corresponds to $7.5 \times 10^{21}\, h^{-1}\, M_\odot\, D_H^{-3}$, where $M_\odot$ is the mass of the Sun.

Most theorists believe that it is in some sense "unlikely" that all three of these density parameters be of the same order, and we know that Ω_M is significantly larger than zero, so many guess that $(\Omega_M, \Omega_\Lambda, \Omega_R) = (1, 0, 0)$, with $(\Omega_M, 1 - \Omega_M, 0)$ and $(\Omega_M, 0, 1 - \Omega_M)$ tied for second place. If $\Omega_\Lambda = 0$, then the *deceleration parameter* q_0 is just half Ω_M, otherwise q_0 is not such a useful parameter. When I perform cosmographic calculations and I want to cover all the bases, I use the three world models

name	Ω_M	Ω_Λ
Einstein-de Sitter	1	0
low density	0.05	0
high lambda	0.2	0.8

These three models push the observational limits.

3 Redshift

The *redshift* z of an object is the fractional doppler shift of its emitted light resulting from radial motion

$$z \equiv \frac{\nu_e}{\nu_o} - 1 = \frac{\lambda_o}{\lambda_e} - 1 \tag{8}$$

where ν_o and λ_o are the observed frequency and wavelength, and ν_e and λ_e are the emitted. Redshift is related to radial velocity v by

$$1 + z = \sqrt{\frac{1 + v/c}{1 - v/c}} \tag{9}$$

where c is the speed of light. The difference between an object's measured redshift and its *cosmological redshift* is due to its (radial) *peculiar velocity;* i.e., we define the cosmological redshift as that part of the redshift due solely to the expansion of the Universe, or *Hubble flow.* In terms of cosmography, the cosmological redshift is directly related to the scale factor $a(t)$, or the "size" of the Universe. For an object at redshift z

$$1 + z = \frac{a(t_o)}{a(t_e)} \tag{10}$$

where $a(t_o)$ is the size of the Universe at the time the light from the object is observed, and $a(t_e)$ is the size at the time it was emitted.

For small v/c, or small distance d, in the expanding Universe, the velocity is proportional to the distance (and all the distance measures, e.g., angular diameter distance, luminosity distance, etc., converge); taking the linear approximation this reduces to

$$z \approx \frac{v}{c} = \frac{d}{D_H} \tag{11}$$

where D_H is the Hubble distance (see above). But this is *only true for small redshifts!*

Redshift is almost always determined with respect to us (or the frame centered on us but stationary with respect to the microwave background), but it is possible to define the redshift z_{12} between objects 1 and 2, both of which are cosmologically redshifted relative to us: the redshift z_{12} of an object at redshift z_2 relative to a hypothetical observer at redshift $z_1 < z_2$ is given by

$$1 + z_{12} = \frac{a(t_1)}{a(t_2)} = \frac{1 + z_2}{1 + z_1} \tag{12}$$

4 Comoving distance (line-of-sight)

A small *comoving distance* δD_C between two nearby objects in the Universe is the distance between them which remains constant with epoch if the two objects are moving with the Hubble flow. In other words, it is the distance between them which would be measured with rulers at the time they are being observed (the *proper distance*) divided by the ratio of the scale factor of the Universe then to now. In other words the proper distance multiplied by $(1 + z)$. The total line-of-sight comoving distance D_C from us to a distant object is computed by integrating the infinitesimal δD_C contributions between nearby events along the radial ray from $z = 0$ to the object.

Following Peebles (1993, pp. 310–321) (who calls the transverse comoving distance by the very confusing name "angular size distance," which is *not* the same as "angular diameter distance" introduced below), we define the function

$$E(z) \equiv \sqrt{\Omega_M\,(1+z)^3 + \Omega_R\,(1+z)^2 + \Omega_\Lambda} \tag{13}$$

which is proportional to the time derivative of the logarithm of the scale factor (i.e., $\dot{a}(t)/a(t)$), with z redshift and the three density parameters defined as above. Since $dz = da$, $dz/E(z)$ is proportional to the time-of-flight of a photon traveling across the redshift interval dz, divided by the scale factor at that time. Since the speed of light is constant, this is a proper distance divided by the scale factor, which is the definition of a comoving distance. The total line-of-sight comoving distance is then given by integrating these contributions, or

$$D_C = D_H \int_0^z \frac{dz'}{E(z')} \tag{14}$$

where D_H is the Hubble distance defined above.

In some sense the line-of-sight comoving distance is the fundamental distance measure in cosmography since, as will be seen below, all others are quite simply derived in terms of it. The line-of-sight comoving distance between two nearby events (i.e., close in redshift or distance) is the distance which we would measure locally between the events today if those two points were locked into the Hubble flow. It is the correct distance measure for measuring aspects of large-scale structure imprinted on the Hubble flow, e.g., distances between "walls."

5 Comoving distance (transverse)

The comoving distance between two events at the same redshift or distance but separated on the sky by some angle $\delta\theta$ is $D_M\,\delta\theta$ and the transverse comoving distance D_M (so-denoted for a reason explained below) is simply related to the line-of-sight comoving distance D_C:

$$D_M = \begin{cases} D_H \frac{1}{\sqrt{\Omega_R}} \sinh\left[\sqrt{\Omega_R}\,D_C/D_H\right] & \text{for } \Omega_R > 0 \\ D_C & \text{for } \Omega_R = 0 \\ D_H \frac{1}{\sqrt{|\Omega_R|}} \sin\left[\sqrt{|\Omega_R|}\,D_C/D_H\right] & \text{for } \Omega_R < 0 \end{cases} \tag{15}$$

where the trigonometric functions sinh and sin account for what is called "the curvature of space." (Space curvature depends on the particular coordinate system chosen, so it is not intrinsic; a change of coordinates makes space flat; the only intrinsic curvature is space-time curvature, which is related to the local mass-energy density or really stress-energy tensor.) For $\Omega_\Lambda = 0$, there is an analytic solution to the equations

$$D_M = D_H \frac{2\left[2 - \Omega_M\,(1-z) - (2-\Omega_M)\,\sqrt{1+\Omega_M\,z}\right]}{\Omega_M^2\,(1+z)} \quad \text{for } \Omega_\Lambda = 0 \tag{16}$$

(Weinberg, 1972, p. 485; Peebles, 1993, pp. 320–321). Weedman (1986, pp. 59–60) calls this distance measure "proper distance," which is very bad style[1], and gives the above formula, also for $\Omega_\Lambda = 0$ but in terms of $q_0 = \Omega_M/2$.

(Although these notes follow the Peebles derivation, there is a qualitatively distinct method using what is known as the *development angle* χ, which increases as the Universe evolves. This method is preferred by relativists such as Misner, Thorne & Wheeler 1973, pp. 782–785).

The comoving distance happens to be equivalent to the *proper motion distance* (hence the name D_M), defined as the ratio of the actual transverse velocity (in distance over time) of an object to its proper motion (in radians per unit time) (Weinberg, 1972, pp. 423–424). The proper motion distance is plotted in Figure 1. Proper motion distance is used, for example, in computing radio jet velocities from knot motion.

[1] The word "proper" has a specific use in relativity. The *proper time* between two nearby events is the time delay between the events in the frame in which they take place at the same location, and the *proper distance* between two nearby events is the distance between them in the frame in which they happen at the same time. It is the distance measured by a ruler at the time of observation. The transverse comoving distance D_M is *not* a proper distance—it is a proper distance divided by a ratio of scale factors.

6 Angular diameter distance

The *angular diameter distance* D_A is defined as the ratio of an object's physical transverse size to its angular size (in radians). It is used to convert angular separations in telescope images into proper separations at the source. It is famous for not increasing indefinitely as $z \to \infty$; it turns over at $z \sim 1$ and thereafter more distant objects actually appear larger in angular size. Angular diameter distance is related to the transverse comoving distance by

$$D_A = \frac{D_M}{1+z} \tag{17}$$

(Weinberg, 1972, pp. 421–424; Weedman, 1986, pp. 65–67; Peebles, 1993, pp. 325–327). The angular diameter distance is plotted in Figure 2.

There is also an angular diameter distance D_{A12} between two objects at redshifts z_1 and z_2, frequently used in gravitational lensing. It is *not* found by subtracting the two individual angular diameter distances! The correct formula, for $\Omega_R \geq 0$, is

$$D_{A12} = \frac{1}{1+z_2}\left[D_{M2}\sqrt{1+\Omega_R\,\frac{D_{M1}^2}{D_H^2}} - D_{M1}\sqrt{1+\Omega_R\,\frac{D_{M2}^2}{D_H^2}}\right] \tag{18}$$

where D_{M1} and D_{M2} are the transverse comoving distances to z_1 and z_2, D_H is the Hubble distance, and Ω_R is the curvature density parameter (Peebles, 1993, pp. 336–337). Unfortunately, the above formula is *not correct* for $\Omega_R < 0$ (Helbig, private communication). Watch this space for the more general version.

7 Luminosity distance

The *luminosity distance* D_L is defined by the relationship between bolometric (i.e., integrated over all frequencies) flux S and bolometric luminosity L:

$$D_L \equiv \sqrt{\frac{L}{4\pi\,S}} \tag{19}$$

It turns out that this is related to the transverse comoving distance and angular diameter distance by

$$D_L = (1+z)\,D_M = (1+z)^2\,D_A \tag{20}$$

(Weinberg, 1972, pp. 420–424; Weedman, 1986, pp. 60–62). The latter relation follows from the fact that the surface brightness of a receding object is reduced by a factor $(1+z)^{-4}$, and the angular area goes down as D_A^{-2}. The luminosity distance is plotted in Figure 3.

If the concern is not with bolometric quantities but rather with differential flux S_ν and luminosity L_ν, as is usually the case in astronomy, then a correction, the *k-correction,* must be applied to the flux or luminosity because the redshifted object is emitting flux in a different band than that in which you are observing. The k-correction depends on the spectrum of the object in question, and is unnecessary only if the object has spectrum $\nu\,L_\nu =$ constant. For any other spectrum the differential flux S_ν is related to the differential luminosity L_ν by

$$S_\nu = (1+z)\,\frac{L_{(1+z)\nu}}{L_\nu}\,\frac{L_\nu}{4\pi\,D_L^2} \tag{21}$$

where z is the redshift, the ratio of luminosities equalizes the difference in flux between the observed and emitted bands, and the factor of $(1+z)$ accounts for the redshifting of the bandwidth. Similarly, for differential flux per unit wavelength,

$$S_\lambda = \frac{1}{(1+z)} \frac{L_{\lambda/(1+z)}}{L_\lambda} \frac{L_\lambda}{4\pi\, D_L^2} \tag{22}$$

(Peebles, 1993, pp. 330–331; Weedman, 1986, pp. 60–62). In this author's opinion, the most natural flux unit is differential flux per unit log frequency or log wavelength $\nu\, S_\nu = \lambda\, S_\lambda$ for which there is no redshifting of the bandpass so

$$\nu\, S_\nu = \frac{\nu_e\, L_{\nu_e}}{4\pi\, D_L^2} \tag{23}$$

where $\nu_e = (1+z)\nu$ is the emitted frequency.

The *distance modulus* DM is defined by

$$DM \equiv 5\, \log\left(\frac{D_L}{10\text{ pc}}\right) \tag{24}$$

because it is the magnitude difference between an object's observed bolometric flux and what it would be if it were at 10 pc (don't ask me, ask an astronomer!). The distance modulus is plotted in Figure 4. The absolute magnitude M is the astronomer's measure of luminosity, defined to be the apparent magnitude the object in question would have if it were at 10 pc, so

$$m = M + DM + K \tag{25}$$

where K is the k-correction

$$K = -2.5\, \log\left[(1+z)\, \frac{L_{(1+z)\nu}}{L_\nu}\right] = -2.5\, \log\left[\frac{1}{(1+z)} \frac{L_{\lambda/(1+z)}}{L_\lambda}\right] \tag{26}$$

8 Parallax distance

If it was possible to measure parallaxes for high redshift objects, the distance so measured would be the *parallax distance* D_P (Weinberg, 1972, pp. 418–420). It may be possible, one day, to measure parallaxes to distant galaxies using gravitational lensing, although in these cases, a modified parallax distance is used which takes into account the redshifts of both the source and the lens (Schneider, Ehlers & Falco, 1992, pp. 508–509), a discussion of which is beyond the scope of these notes.

9 Comoving volume

The *comoving volume* V_C is the volume measure in which number densities of non-evolving objects locked into Hubble flow are constant with redshift. It is the proper volume times three factors of the relative scale factor now to then, or $(1+z)^3$. Since the derivative of comoving distance with redshift is $1/E(z)$ (defined above), the angular diameter distance converts a solid angle $d\Omega$ into a proper area, and two factors of $(1+z)$ convert a proper area into a comoving area, the comoving volume element in solid angle $d\Omega$ and redshift interval dz is

$$dV_C = D_H\, \frac{(1+z)^2\, D_A^2}{E(z)}\, d\Omega\, dz \tag{27}$$

where D_A is the angular diameter distance at redshift z and $E(z)$ is defined above (Weinberg, 1972, p. 486; Peebles, 1993, pp. 331–333). The comoving volume element is plotted in Figure 5. The comoving volume element and its integral are both used frequently in predicting number counts or luminosity densities.

10 Lookback time

The *lookback time* t_L to an object is the difference between the age t_o of the Universe now (at observation) and the age t_e of the Universe at the time the photons were emitted (according to the object). It is used to predict properties of high-redshift objects with evolutionary models, such as passive stellar evolution for galaxies. Recall that $E(z)$ is the time derivative of the logarithm of the scale factor $a(t)$; the scale factor is proportional to $(1+z)$, so the product $(1+z)\,E(z)$ is proportional to the derivative of z with respect to the lookback time, or

$$t_L = t_H \int_0^z \frac{dz'}{(1+z')\,E(z')} \tag{28}$$

(Peebles, 1993, pp. 313–315; Kolb & Turner 1990, pp. 52–56, give some analytic solutions to this equation, but they are concerned with the age $t(z)$, so they integrate from z to ∞). The lookback time is plotted in Figure 6.

11 Probability of intersecting objects

Given a population of objects with comoving number density $n(z)$ (number per unit volume) and cross section $\sigma(z)$ (area), what is the incremental probability dP that a line of sight will intersect one of the objects in redshift interval dz at redshift z? Questions of this form are asked frequently in the study of QSO absorption lines or pencil-beam redshift surveys. The answer is

$$dP = n(z)\,\sigma(z)\,D_H\,\frac{(1+z)^2}{E(z)}\,dz \tag{29}$$

(Peebles, 1993, pp. 323–325). The dimensionless differential intersection probability is plotted in Figure 7.

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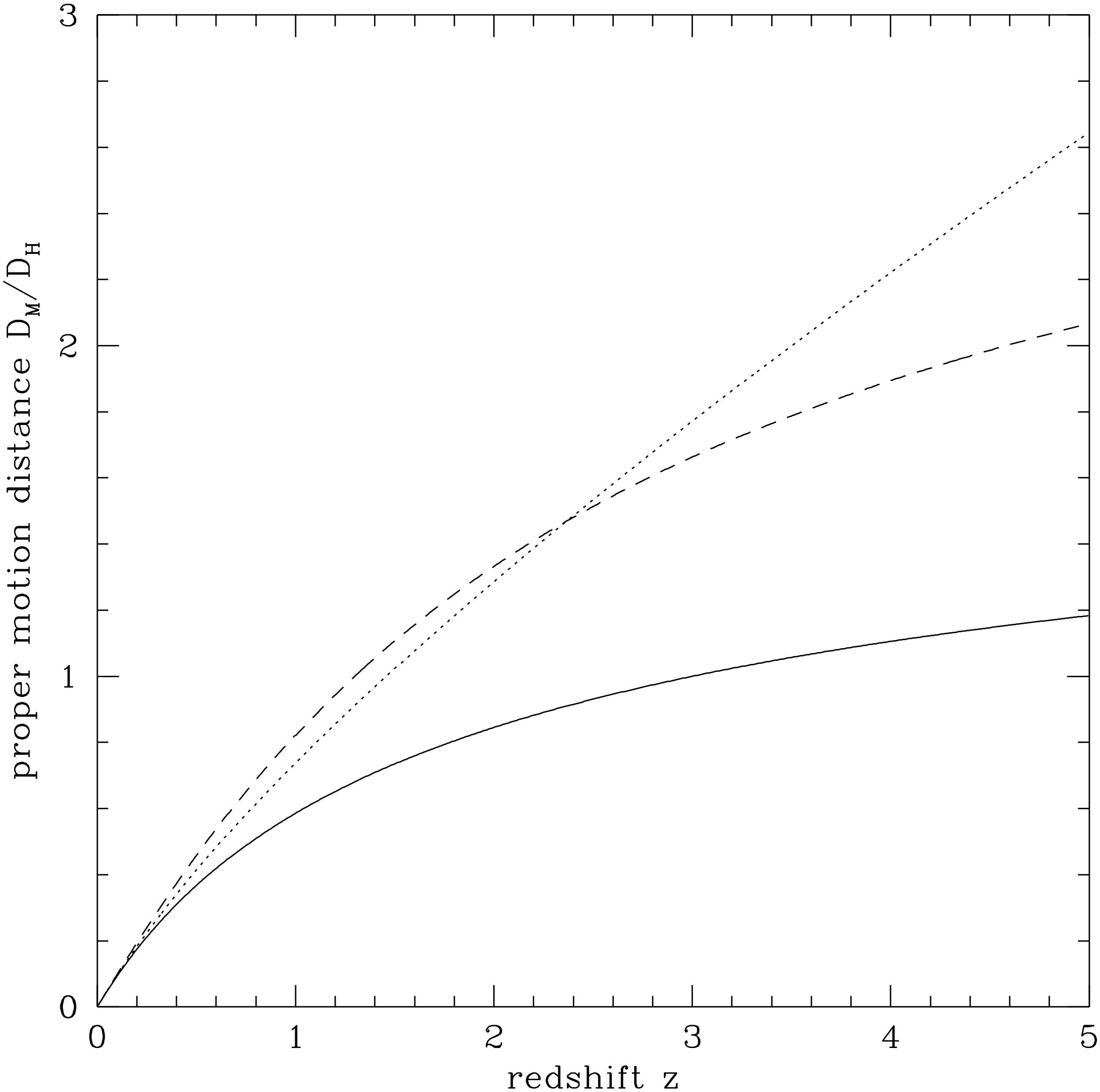

Figure 1: The dimensionless proper motion distance D_M/D_H. The three curves are for the three world models, Einstein-de Sitter $(\Omega_M, \Omega_\Lambda) = (1, 0)$, solid; low-density, $(0.05, 0)$, dotted; and high lambda, $(0.2, 0.8)$, dashed.

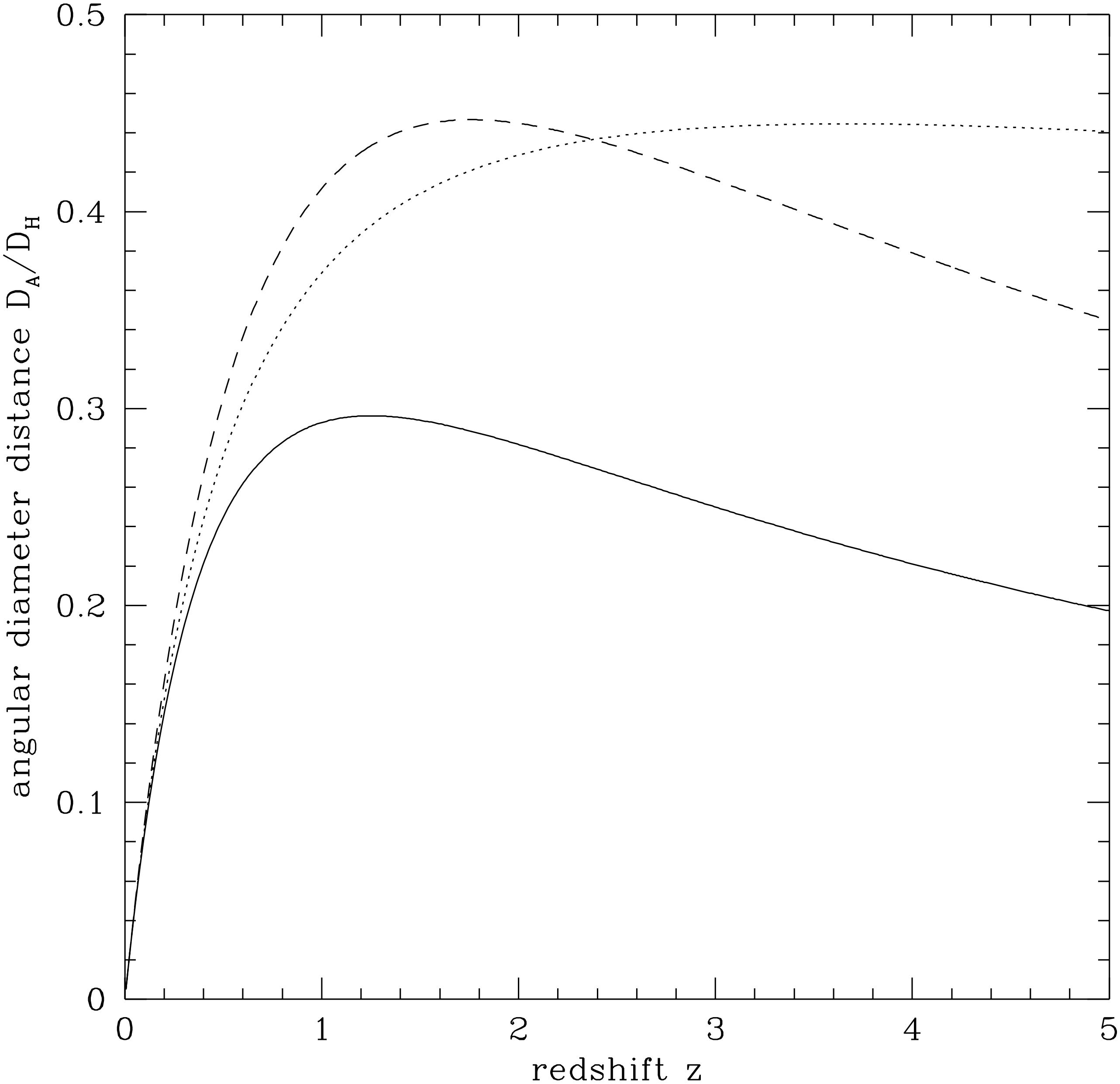

Figure 2: The dimensionless angular diameter distance D_A/D_H. The three curves are for the three world models, $(\Omega_M, \Omega_\Lambda) = (1, 0)$, solid; $(0.05, 0)$, dotted; and $(0.2, 0.8)$, dashed.

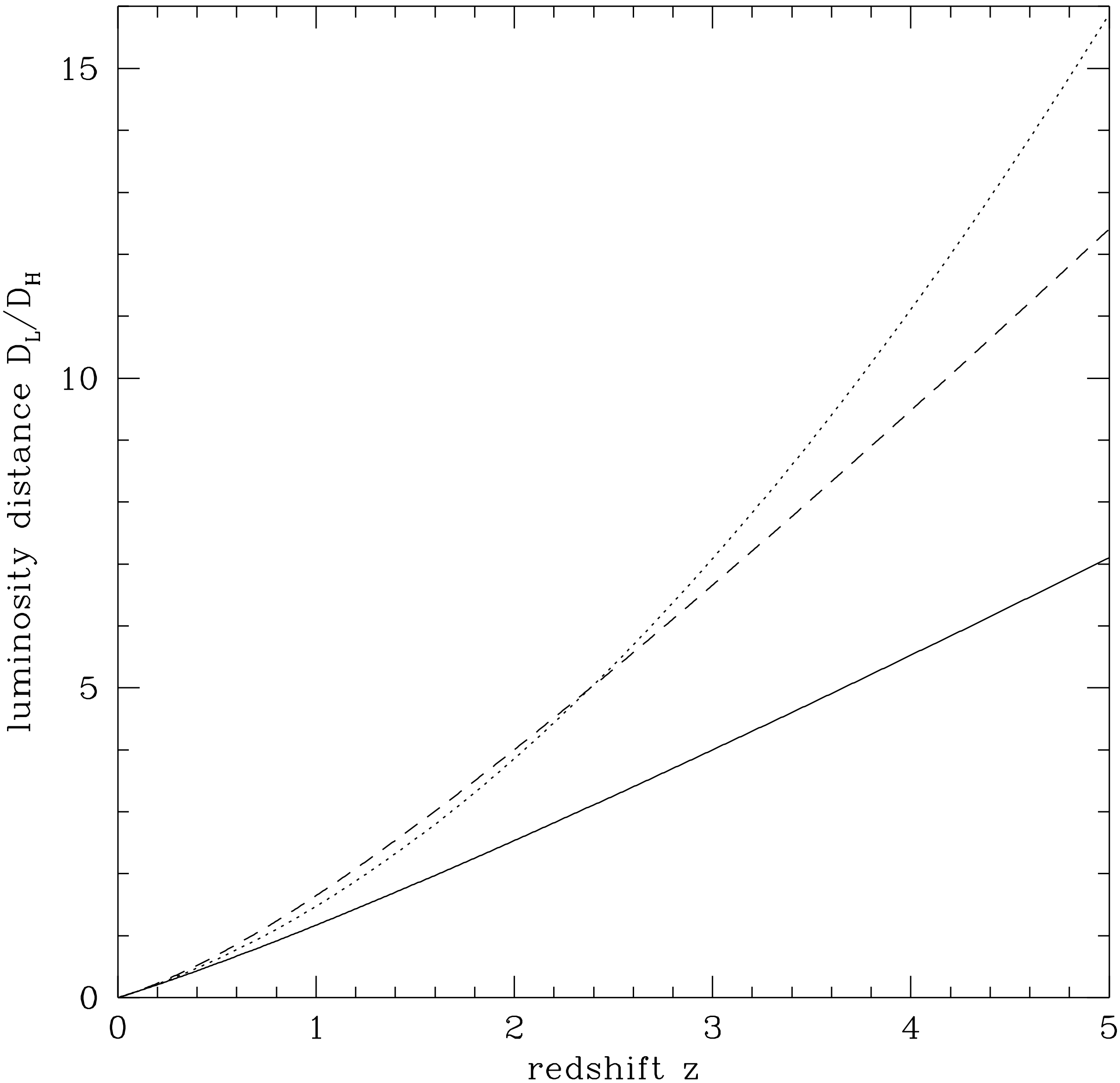

Figure 3: The dimensionless luminosity distance D_L/D_H. The three curves are for the three world models, $(\Omega_M, \Omega_\Lambda) = (1, 0)$, solid; $(0.05, 0)$, dotted; and $(0.2, 0.8)$, dashed.

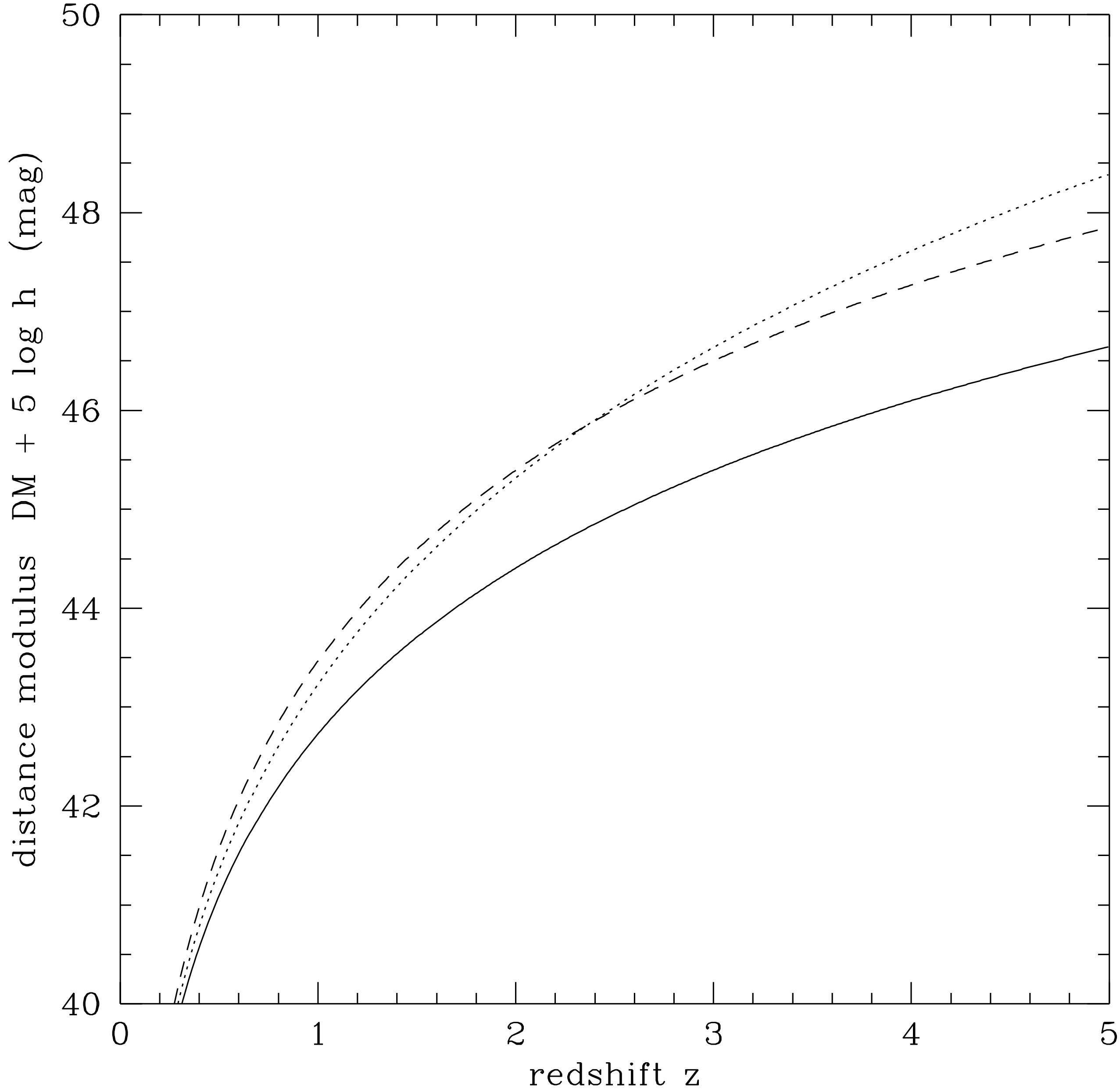

Figure 4: The distance modulus DM. The three curves are for the three world models, $(\Omega_M, \Omega_\Lambda) = (1, 0)$, solid; $(0.05, 0)$, dotted; and $(0.2, 0.8)$, dashed.

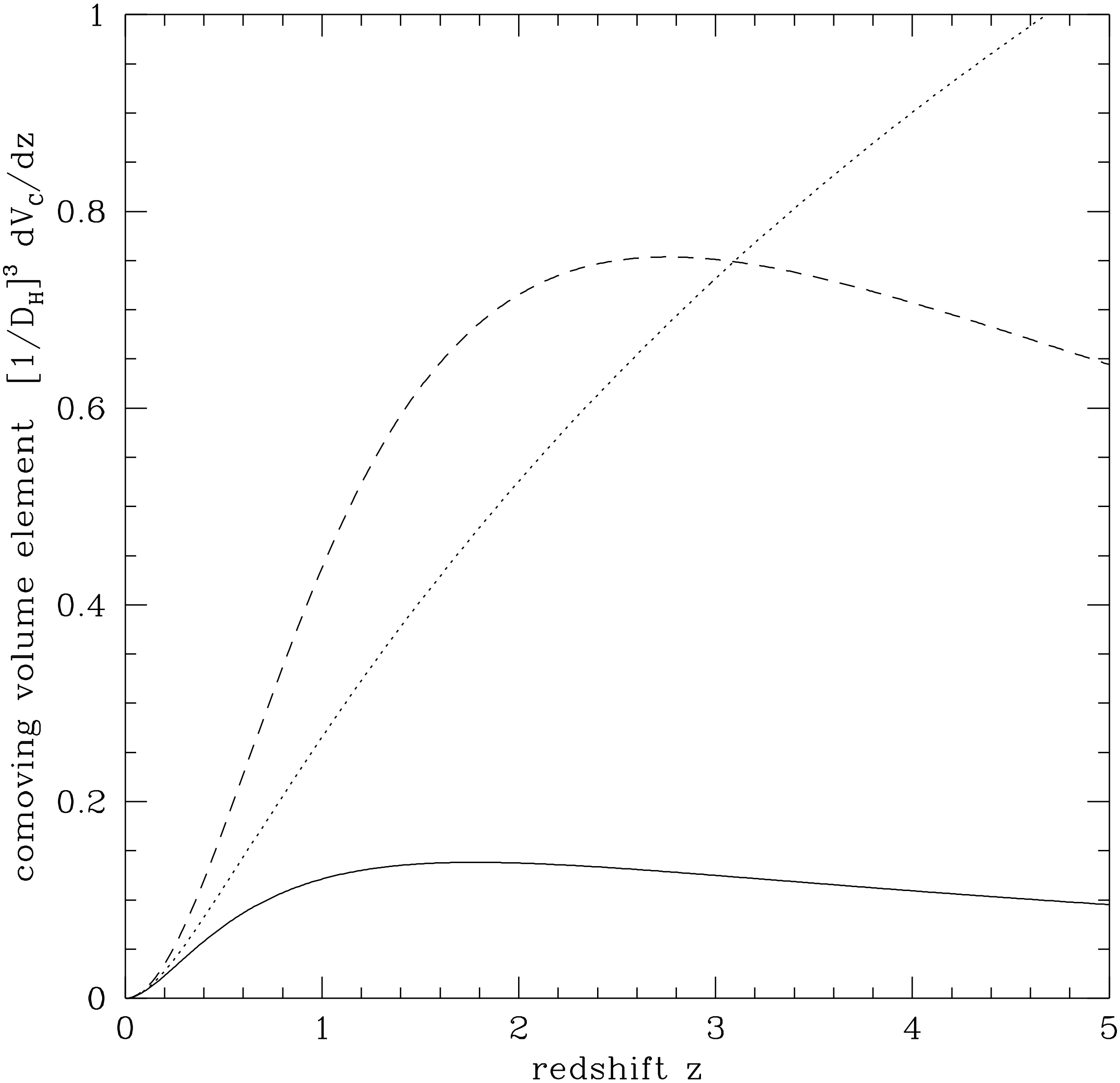

Figure 5: The dimensionless comoving volume element $(1/D_H)^3\,(dV_C/dz)$. The three curves are for the three world models, $(\Omega_M, \Omega_\Lambda) = (1, 0)$, solid; $(0.05, 0)$, dotted; and $(0.2, 0.8)$, dashed.

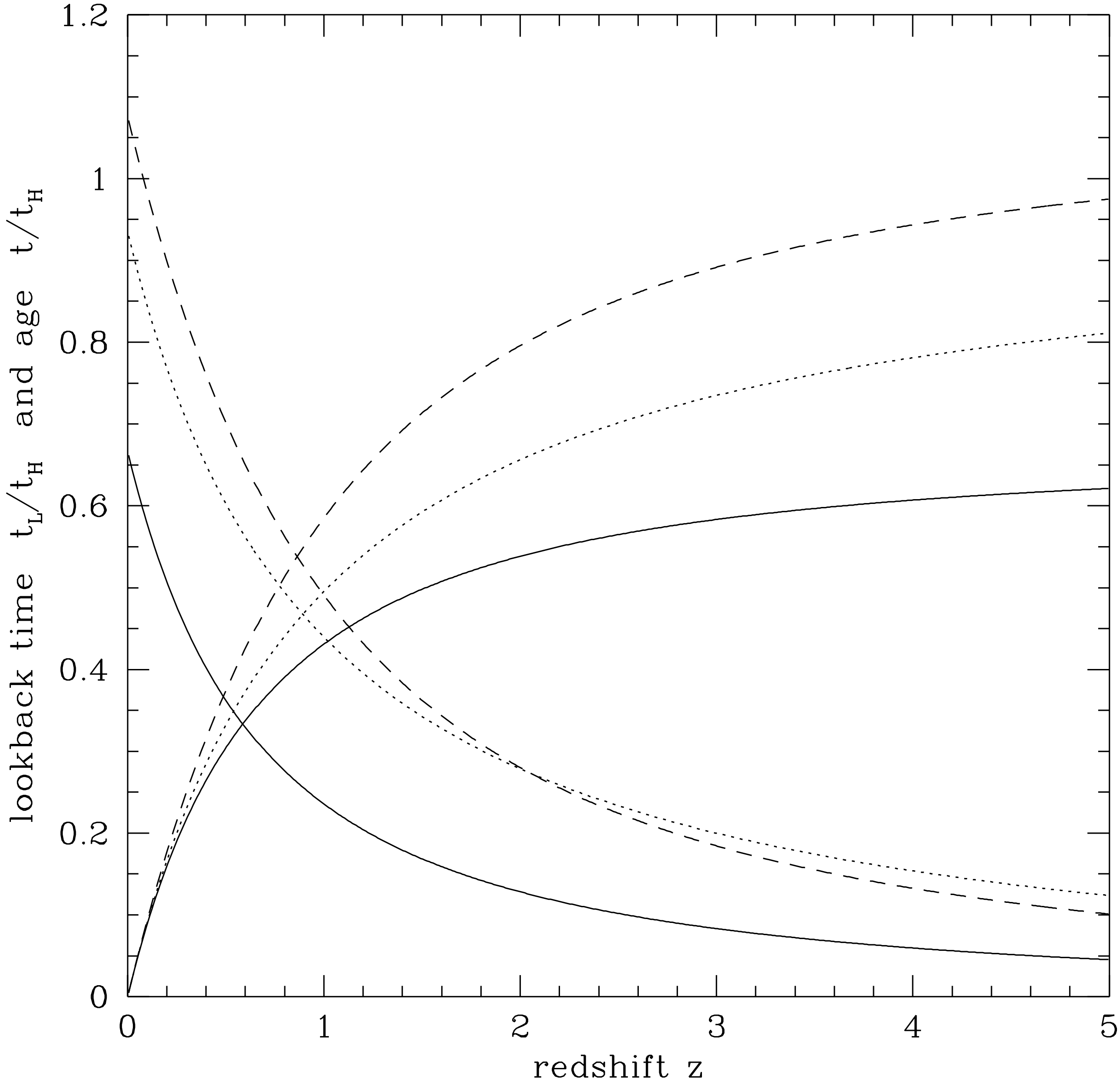

Figure 6: The dimensionless lookback time t_L/t_H and age t/t_H. Curves cross at the redshift at which the Universe is half its present age. The three curves are for the three world models, $(\Omega_M, \Omega_\Lambda) = (1, 0)$, solid; $(0.05, 0)$, dotted; and $(0.2, 0.8)$, dashed.

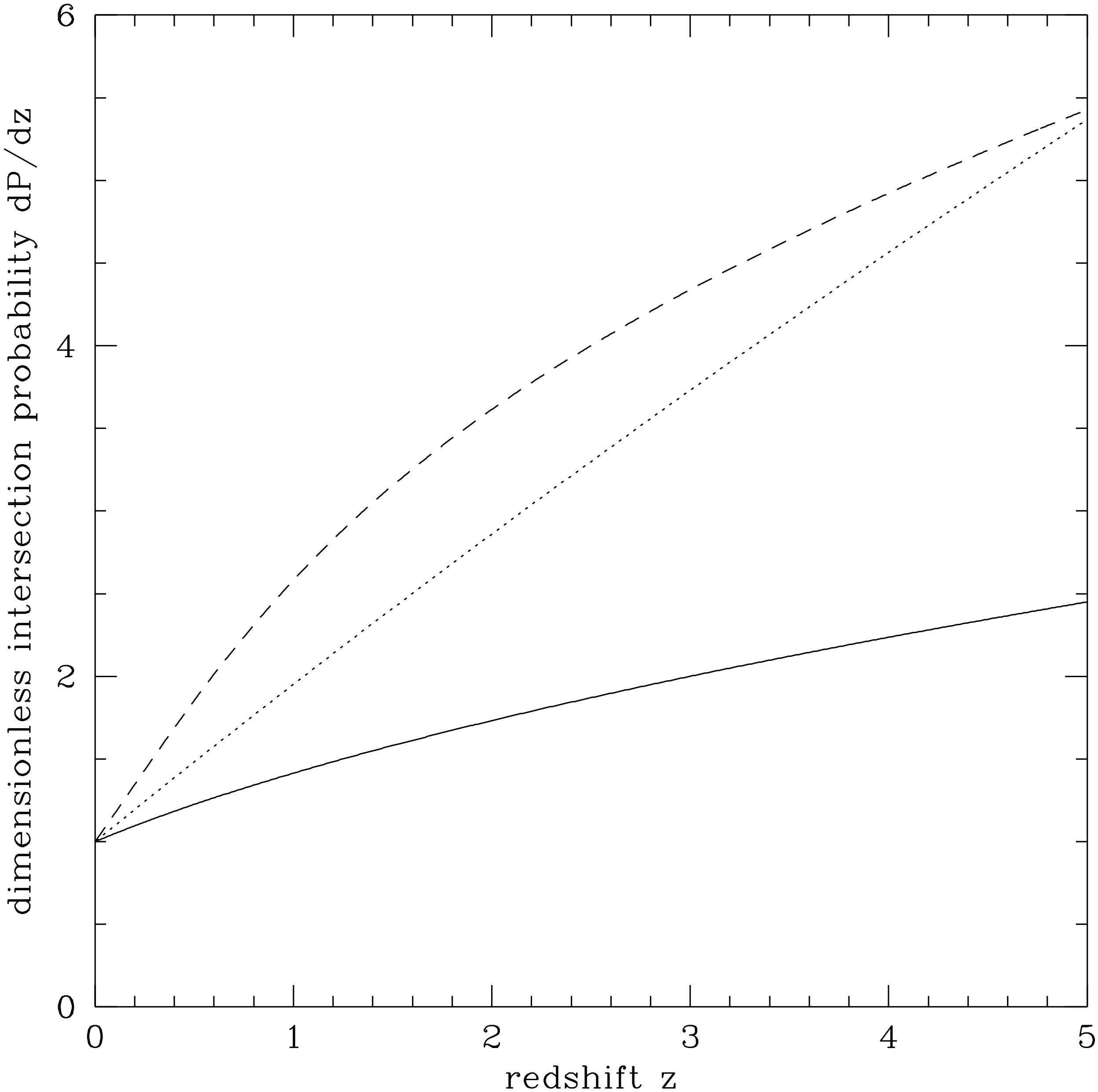

Figure 7: The dimensionless differential intersection probability dP/dz; dimensionless in the sense of $n(z)\,\sigma(z)\,D_H = 1$. The three curves are for the three world models, $(\Omega_M, \Omega_\Lambda) = (1, 0)$, solid; $(0.05, 0)$, dotted; and $(0.2, 0.8)$, dashed.